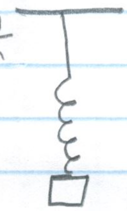
3.8: Forced Vibrations with Damping

CONSIDER $mu'' + \gamma u' + ku = F_0 \cos(\omega t)$, WHERE $\gamma > 0$.

ENTRY TASK $m = 1\ kg$, $\gamma = 2\ \frac{N}{m/s}$, $k = 5\ \frac{N}{m}$



Solve

$$u'' + 2u' + 5u = 10\cos(t) \qquad \boxed{\omega = 1}$$

1. $r^2 + 2r + 5 = 0$

$$r = \frac{-2 \pm \sqrt{2^2 - 4(5)}}{2} = -1 \pm \tfrac{1}{2}\sqrt{-16} = -1 \pm 2i$$

λ μ

$y_1(t) = e^{-t}\cos(2t)$ $y_2(t) = e^{-t}\sin(2t)$

2. $U(t) = A\cos(t) + B\sin(t)$

$U'(t) = -A\sin(t) + B\cos(t)$

$U''(t) = -A\cos(t) - B\sin(t)$

$$u'' + 2u' + 5u = 10\cos(t)$$

$$-A\cos(t) - B\sin(t) - 2A\sin(t) + 2B\cos(t) + 5A\cos(t) + 5B\sin(t) = 10\cos(t)$$

$$(4A + 2B)\cos(t) + (-2A + 4B)\sin(t) \stackrel{?}{=} 10\cos(t)$$

$\left.\begin{array}{l} 4A + 2B = 10 \\ -2A + 4B = 0 \end{array}\right\}$ $A = 2$, $B = 1$

$\omega = 1$

$$u(t) = \underbrace{c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)}_{u_c(t)} + \underbrace{2\cos(t) + \sin(t)}_{U(t)}$$

$u_c(t)$ transient sol'n, $\mu = 2$

$U(t)$ steady state sol'n (forced response)

SHOW PICTURE

AMPLITUDE

$R = \sqrt{2^2 + 1^2} = \sqrt{5} \approx 2.2361$

$T = \frac{2\pi}{\omega}$

$\omega = 1$

$$u'' + 0.1u' + 5u = 10\cos(t)$$

[1] $r = -\frac{0.1}{2} \pm \frac{1}{2}\sqrt{0.1^2 - 20} = -0.05 \pm \frac{\sqrt{19.99}}{2} i$

λ μ CLOSE TO $\sqrt{5}$

$$u_c(t) = e^{-0.05t}(c_1 \cos(\mu t) + c_2 \sin(\mu t))$$

[2] $U(t) = A\cos(t) + B\sin(t)$

$\Downarrow$

$$(4A + 0.1B)\cos(t) + (-0.1A + 4B)\sin(t) = 10\cos(t)$$

$$\left.\begin{aligned} 4A + 0.1B &= 10 \\ -0.1A + 4B &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= \frac{40}{16.01} \\ B &= \frac{1}{16.01} \end{aligned}$$

$$U(t) = \frac{40}{16.01}\cos(t) + \frac{1}{16.01}\sin(t)$$

$\gamma = 0.1 \Rightarrow$ TRANSIENT SOL'N "LASTS" LONGER

$R = 1.250781$

$$u'' + \gamma u' + 5u = 10\cos(\sqrt{5}t)$$

$\omega = \sqrt{5}$ ← FORCING FREQUENCY

$r = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2 - 20}$

$\lambda = -\frac{\gamma}{2}$

$\mu = \frac{1}{2}\sqrt{20 - \gamma^2}$ ← QUASIFREQUENCY

$\gamma = 0$

$\omega_0 = \sqrt{5}$ ← natural frequency

$$U(t) = A\cos(\sqrt{5}t) + B\sin(\sqrt{5}t)$$

$$U'(t) = -\sqrt{5}A\sin(\sqrt{5}t) + \sqrt{5}B\cos(\sqrt{5}t)$$

$$U''(t) = -5A\cos(\sqrt{5}t) - 5B\sin(\sqrt{5}t)$$

$u'' + 5u = 0$!

$$-\gamma\sqrt{5}A\sin(\sqrt{5}t) + \gamma\sqrt{5}B\cos(\sqrt{5}t) = 10\cos(\sqrt{5}t)$$

$A = 0$

$B = \frac{10}{\gamma\sqrt{5}} = \frac{2\sqrt{5}}{\gamma}$

Graphs of sol'ns to

$$u'' + 2u' + 5u = 10\cos(t)$$

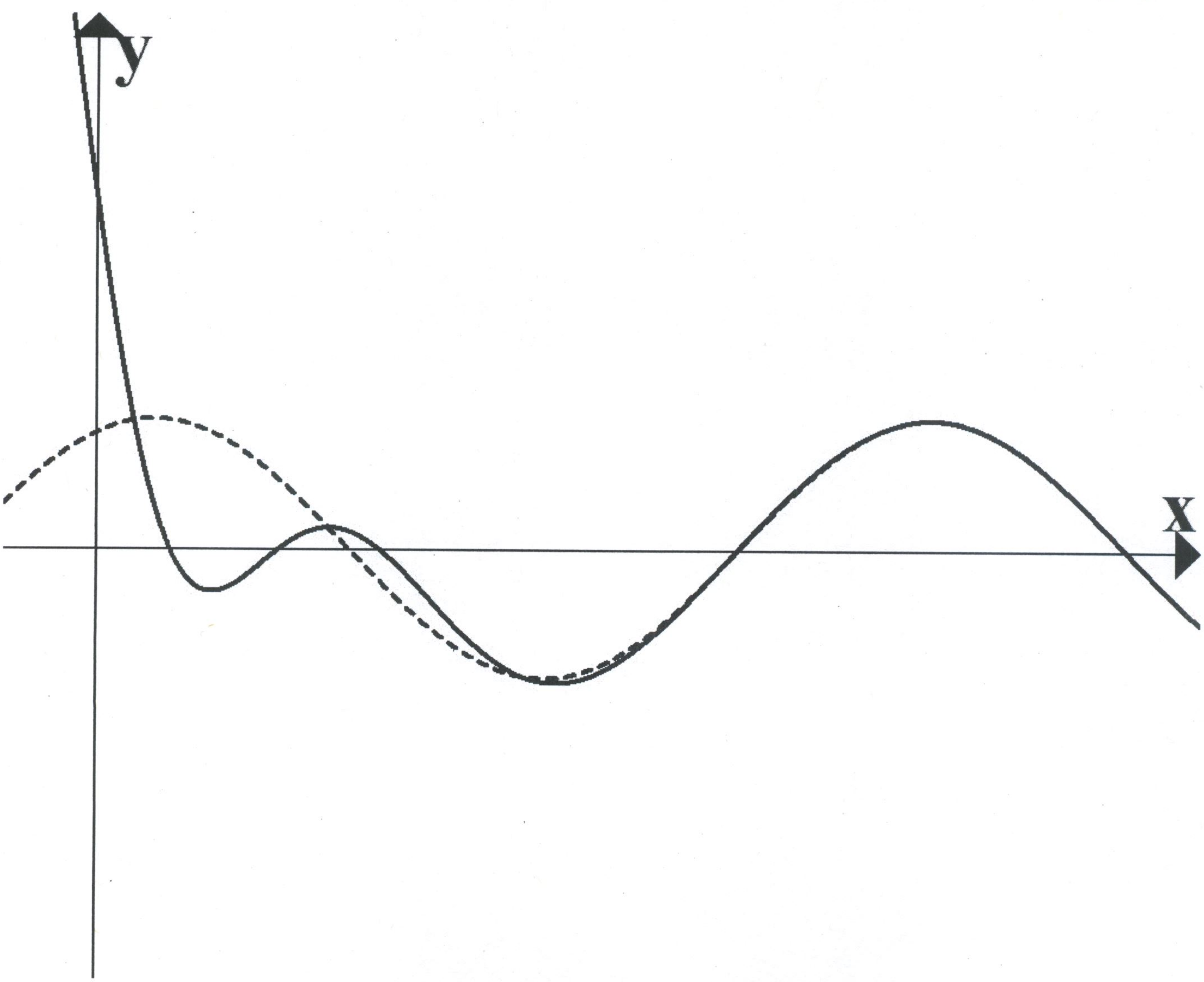
Steady state solution (dotted):

$$U(t) = 2\cos(t) + \sin(t)$$

The solution with initial conditions

u(0) = 6 and u'(0) = -11 is shown. This solution is:

$$u(t) = e^{-t}(4\cos(2\mathrm{t}) - 6\sin(2t)) + U(t)$$

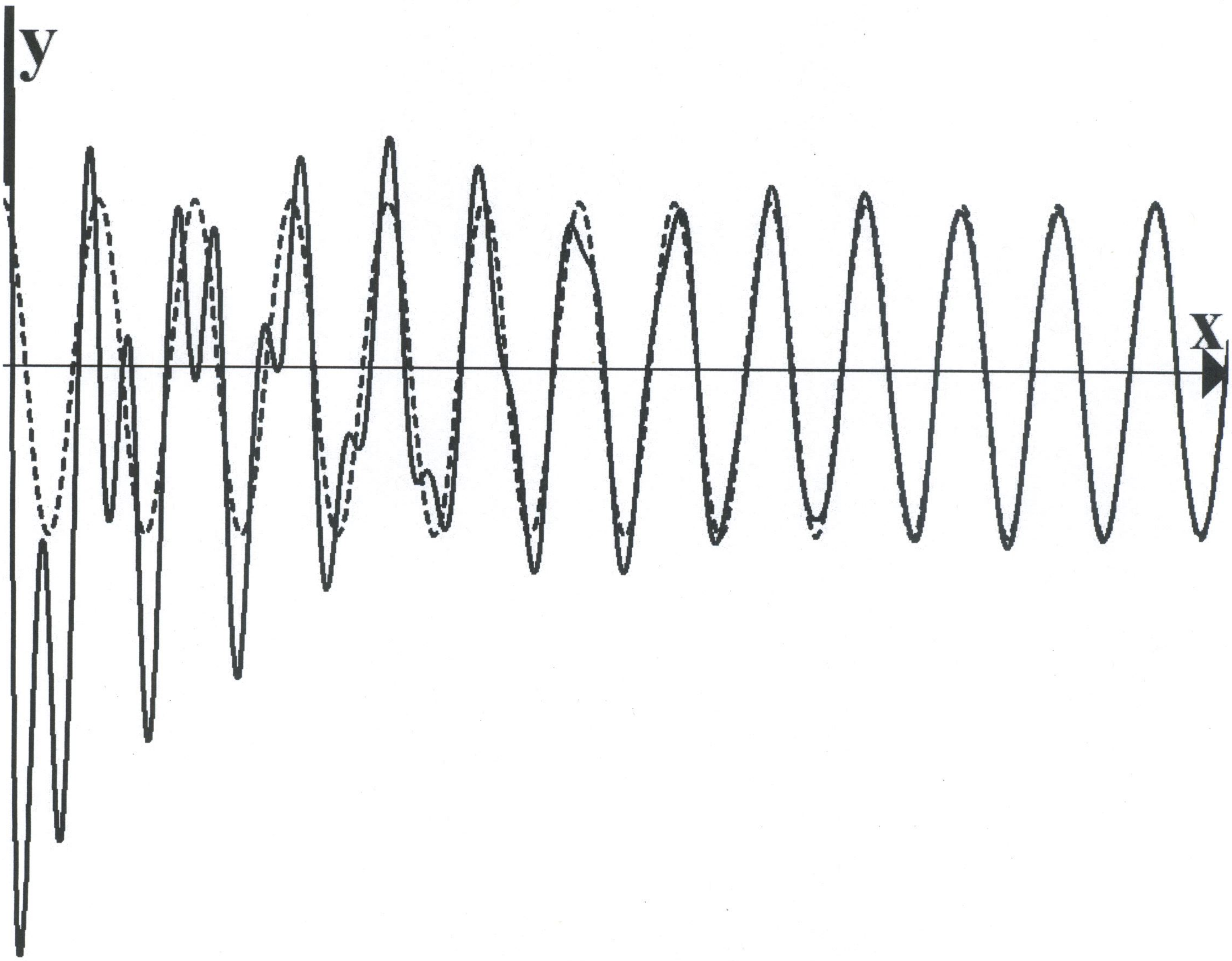


Graphs of sol'ns to

$$u'' + 0.1u' + 5u = 10\cos(t)$$

Steady state solution:

$$U(t) = \frac{40}{16.01}\cos(t) + \frac{1}{16.01}\sin(t)$$

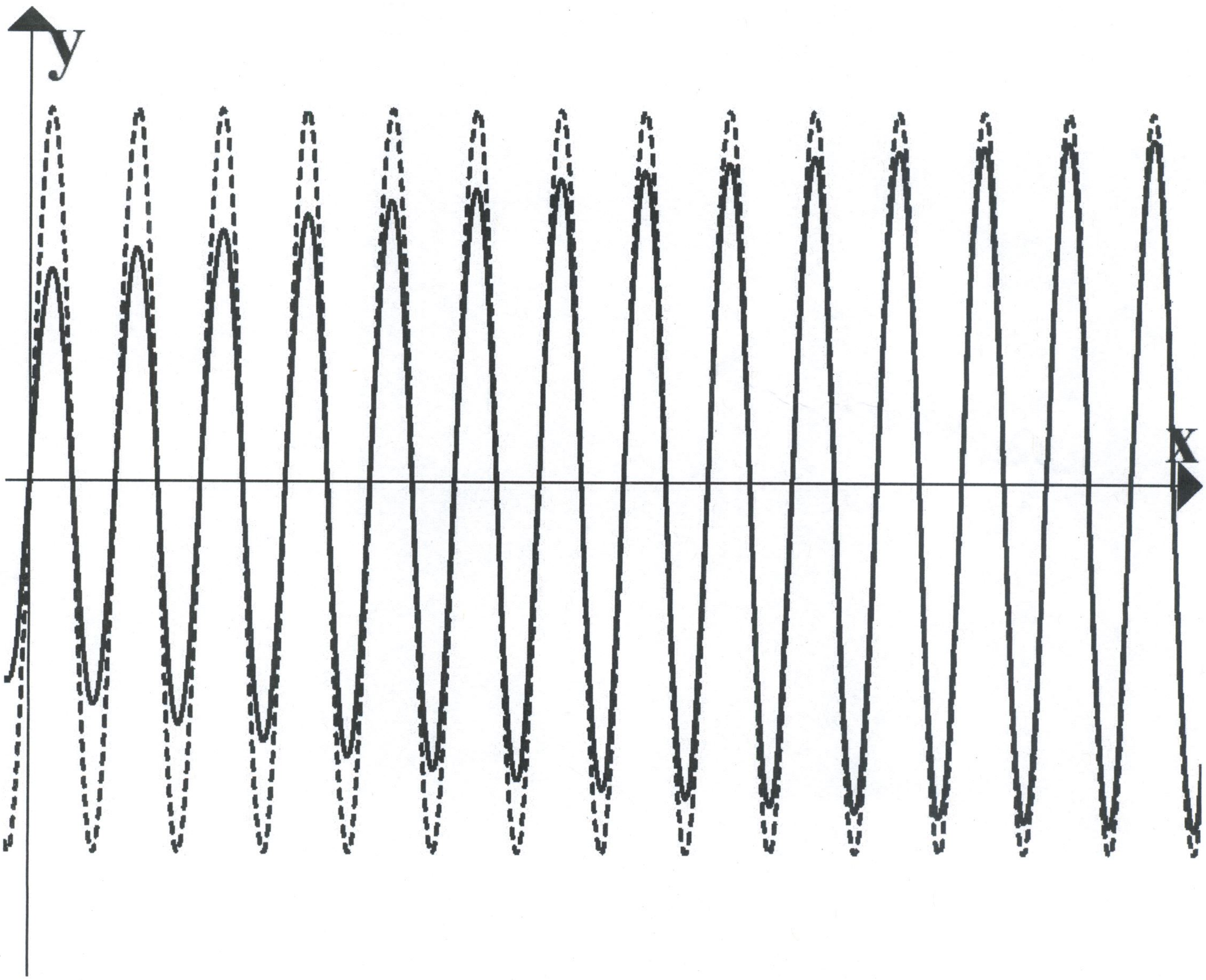


Graphs of sol'ns to

$$u'' + 0.1u' + 5u = 10\cos(\sqrt{5}\,t)$$

Steady state solution:

$$U(t) = \frac{2\sqrt{5}}{0.1}\sin(\sqrt{5}\,t) \approx 44.72\sin(\sqrt{5}\,t)$$



Consider

$$u'' + \gamma u' + 5u = 10\cos(\sqrt{5}\,t)$$

Steady state solution:

$$U(t) = \frac{2\sqrt{5}}{\gamma}\sin(\sqrt{5}\,t)$$

γ	Amplitude of the steady state solution
10	0.447
1	4.47
0.1	44.72
0.01	447.21
0.001	4472.14

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

Note: $\omega_0 = \sqrt{\frac{k}{m}}$, $\mu = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$, $\lambda = -\frac{\gamma}{2m}$

Particular Solution:

$$U(t) = A\cos(\omega t) + B\sin(\omega t)$$

Leads to:

$$-\gamma\omega A + (k - m\omega^2)B = 0$$

$$(k - m\omega^2)A + \gamma\omega B = \mathrm{F}_0$$

$$R = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}}$$

If $\omega \approx \omega_0$, then $R \approx \frac{F_0}{\gamma\omega}$. (Resonance)

For small values of γ, this will be the maximum amplitude.

Aside: $\omega_{max} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}}$

If there is NO damping we get:

$$u'' + 5u = 10\cos(\sqrt{5}\,t)$$

General Solution:

$$u(t) = c_1 \cos\left(\sqrt{5}\,t\right) + c_2 \sin\left(\sqrt{5}\,t\right) + \sqrt{5}t\sin(\sqrt{5}\,t)$$